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THE MODULI SPACE OF ENRIQUES SURFACES AND THE FAKE MONSTER LIE SUPERALGEBRA

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WE SHOW THAT the moduli space of complex Enriques surfaces is an affine variety with a copy of the affine line removed. We do this by using the denominator function of a generalized Kac–Moody superalgebra (associated with superstrings on a 10-dimensional torus) to construct a non-vanishing section of an ample line bundle on the moduli space. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

The moduli space D^0 of Enriques surfaces is known to be the quotient D of a 10-dimensional hermitian symmetric space Ω by a discrete group $O_M(\mathbf{Z})$, with a divisor H_d removed. The symmetric space Ω has a $O_M(\mathbf{Z})$-line bundle P over it such that the sections of P^n are essentially automorphic forms of weight n. The line bundle P is ample on D and some power defines an embedding of $D = \Omega/O_M(\mathbf{Z})$ into projective space, which makes D, and hence D^0, into a quasiprojective variety. We will prove

THEOREM 1.1. *The moduli space D^0 is a quasiaffine variety.*

To prove this we will show that the ample bundle P^4 is trivial when restricted to the complement of the divisor H_d, so that the trivial bundle over the moduli space is ample and therefore the moduli space is quasiaffine (and not just quasiprojective). Sections of P^4 are essentially the same as automorphic forms of weight 4, so to show that P^4 is trivial we construct an automorphic form Φ (Theorem 3.2) of weight 4 on the hermitian symmetric space Ω which has a zero of order 1 along the divisors H_d and has no other zeros. This automorphic form Φ then defines a trivialization of P^4 restricted to the moduli space D^0.

The function Φ is constructed in [2] as a twisted denominator function of the fake monster Lie algebra, associated to an automorphism of order 2 of the Leech lattice fixing an 8-dimensional subspace. The fact that Φ is an automorphic form should follow from a generalization of the results of [3] from the level 1 case covered there to higher levels. As this generalization has not yet been done, we prove that Φ is an automorphic form (in Section 3) by an *ad hoc* argument using the fact that Φ can be written as either an infinite product or an infinite sum. We then show that Φ has no zeros other than the hyperplanes H_d.

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It is not hard to describe precisely how D^0 differs from an affine variety: it is an affine variety with a copy of the affine line $\mathbf{C}^1$ removed. This follows from the description of the Baily–Borel compactification of D given by Sterk in [8: 4.5, 4.6, 4.7]. For the readers' convenience we briefly recall Sterk's results. The Baily–Borel compactification consists of D together with 2 points corresponding to the 2 orbits of primitive isotropic vectors in M, and two 1-dimensional pieces isomorphic to $\mathbf{C}$ and $\mathbf{C}^*$ corresponding to the two orbits of primitive isotropic rank 2 sublattices of M. The closure of the divisor H_d of D contains one of the points and the 1-dimensional component $\mathbf{C}$. The complement of the closure of H_d is an affine variety, and this affine variety is just the union of D^0, a copy of $\mathbf{C}^*$, and a point. The copy of $\mathbf{C}^*$ is constructed as the quotient of the upper half plane by the group $\Gamma_0(2) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in SL_2(\mathbf{Z}) | c \equiv 2 \bmod 0\}$ and the point is then just one of the cusps, so the union of $\mathbf{C}^*$ and this point is just a copy of the affine line $\mathbf{C}$. Hence the moduli space D^0 is an affine variety with a copy of the affine line $\mathbf{C}$ removed.

Dolgachev pointed out to me that the form constructed in Theorem 3.2 might be one case of an infinite family of forms as follows. Let R denote one of the following four division algebras $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, $\mathbf{O}$ of real, complex, Hamiltonian, or Cayley numbers. Let $S_n(R)$ denote Hermitian $n \times n$ matrices with entries in R and let $S_n(R)^+$ be the cone of positive-definite matrices. Consider the tube domain $S_n(R) + iS_n(R)^+$. Except when $R = \mathbf{O}$ and $n > 3$ it is a symmetric bounded domain. When $n = 2$, $R = \mathbf{O}$ we get the domain Ω whose quotient D with a divisor H_d removed parameterizes Enriques surfaces. Let $\Gamma(R)$ be the arithmetic group $S_p(2n, \mathbf{Z})$ ($R = \mathbf{R}$), $GL(2n, \mathbf{Z}[i])$ ($R = \mathbf{C}$), $SU(2n, \mathcal{O}_{\mathbf{H}})$ ($R = \mathbf{H}$), $O_M(\mathbf{Z})$ ($R = \mathbf{O}, n = 2$). There might be a similar Γ-modular form on each of these spaces. The complement of its zeroes should be the period space of a family of (possibly non-simply connected) Calabi–Yau manifolds of dimension n. The existence of a such a form is known for $n = 2$, $R \neq \mathbf{H}$. If $R = \mathbf{R}$, $S_2(\mathbf{R}) + iS_2(\mathbf{R})^+$ is the 3-dimensional Siegel space Z_2, and the family is the family of Kummer surfaces. In the case $\mathbf{C}$, the domain $S_2(\mathbf{C}) + iS_2(\mathbf{C})^+$ is 4-dimensional of type $I_{2,2} \cong \mathrm{IV}_4$ and the family is the family of K3-surfaces which are non-singular models of branched covers of the plane ramified over the union of six lines (see [5] for the construction of the corresponding form). The case $R = \mathbf{O}$ is Theorem 3.2. If n is arbitrary and $R = \mathbf{C}$ Dolgachev conjectures that the family is the family of Calabi–Yau n-folds which are obtained by a resolution of double covers of $\mathbf{P}^n$ branched along $2n + 2$ hyperplanes in general position.

Kondo [4] has recently proved that the moduli space of Enriques surfaces is rational.

It is also possible to construct automorphic forms related to the moduli spaces of polarized K3 surfaces using similar methods (as A. Todorov suggested to me). For example, the form constructed in Example 4 of Section 16 of [3] associated with the Dynkin diagram E_7 is an automorphic form for the group $O_{\mathrm{II}_{2,18} \oplus \langle -2 \rangle}(\mathbf{R})$ which vanishes exactly on the hyperplanes of norm -2 vectors, and is very closely related to the moduli space of K3 surfaces with a polarization of degree 2.

NOTATION AND TERMINOLOGY

All varieties are defined over the complex numbers. The bilinear forms on lattices have the opposite signs to those in [3]; this is because the sign conventions in algebraic geometry are the opposite of those in the theory of Lorentzian lattices.

$'$ If M is lattice then M' means the dual of M.

$+$ If G is a subgroup of $O_M(\mathbf{Z})$ then G^+ is the subgroup of G of elements not interchanging the two components of Ω.

$c(n)$ $\sum_n c(n)q^n = \eta(\tau)^{-8}\eta(2\tau)^8\eta(4\tau)^{-8}$.
C An open solid cone in $L \otimes \mathbf{R}$.
$\mathbf{C}$ The complex numbers.
$\Delta(\tau) = \eta(\tau)^{24}$.
d A norm -2 vector of M.
D The complex space $\Omega/O_M(\mathbf{Z})$.
D^0 The moduli space of Enriques surfaces, which is D with the divisor H_d removed.
$E_4(\tau) = 1 + 240\sum_{m>0,n>0} m^3q^{mn}$.
E_8 The E_8 lattice. If n is an integer then $E_8(n)$ means E_8 with the values of the bilinear form multiplied by n.
$\eta(\tau) = q^{1/24}\prod_{n>0}(1-q^n)$.
Φ An automorphic form of weight 4.
$f(\tau) = \eta(\tau)^{-8}\eta(2\tau)^8\eta(4\tau)^{-8}$.
Γ_1 The subgroup of $O_M(\mathbf{Z})$ generated by reflections of R_2 and by -1.
Γ_2 The subgroup of $O_M(\mathbf{Z})$ generated by reflections of $R_0 \cup R_2$ and by -1.
Γ_3 A finite index subgroup of $O_M(\mathbf{Z})$ defined in Lemma 2.5.
H_d The points of Ω which are orthogonal to the norm -2 vector $d \in M$.
$\mathscr{I}(y)$ The imaginary part of y.
$\mathrm{II}_{m,n}$ The even unimodular Lorentzian lattice of dimension $m+n$ and signature $m-n$.
I_5 A modified Bessel function.
L The lattice $E_8(-2) \otimes II_{1,1}$. The element $(v, m, n) \in L$ with $v \in E_8(-2)$, $m, n \in \mathbf{Z}$ has norm $v^2 + 2mn$.
λ A vector in L'.
M The lattice $L \oplus II_{1,1}(2)$. The element $(v, m, n) \in M$ with $v \in L$, $m, n \in \mathbf{Z}$ has norm $v^2 + 4mn$.
μ A vector in L.
m, n Integers.
$O_M(\mathbf{Z})$ The group of all automorphisms of the lattice M.
Π^+ The set of positive vectors of L, i.e., the vectors which have positive inner product with ρ or are positive multiples of ρ.
q $e^{2\pi i\tau}$
$\mathbf{Q}$ The rational numbers.
ρ, ρ' The norm 0 vectors $\rho = (0,0,1)$ and $\rho' = (0,1,0)$ of the lattice $E_8(-2) \oplus II_{1,1}$.
r A norm -2 vector of M.
R_0, R_2 The sets of norm -2 vectors of M which have inner product 0 or 2 with u.
S The surface of points $y \in iC$ with $(y, y) = -1$.
τ A complex number with positive imaginary part.
u The norm 0 vector $(0,0,1) \in L \oplus II_{1,1}(2) = M$.
v A vector of L.
W The reflection group of the lattice $L = E_8(-2) \oplus II_{1,1}$ generated by the reflections of norm -2 vectors.
χ A homomorphism of $O_M(\mathbf{Z})^+$ to $\{\pm 1\}$ taking reflections of norm -2 vectors to -1 and reflections of norm -4 vectors to 1.
y A vector in $L \otimes \mathbf{R} + iC$.
$\Psi(y) = \Phi(y + (\rho - \rho')/2)$.
Ω The hermitian symmetric space (with 2 components) associated with the lattice M, consisting of all points $\omega \in P(M \otimes \mathbf{C})$ such that $(\omega, \omega) = 0$ and $(\omega, \bar{\omega}) > 0$.

2. THE LATTICE M AND THE MODULI SPACE OF ENRIQUES SURFACES

In this section we recall some facts about the moduli space of Enriques surfaces, and the associated lattice and symmetric space. Many of these results can be found in [1, Ch. VIII].

We define the lattice M to be $L \oplus II_{1,1}(2)$, where $L = E_8(-2) \oplus II_{1,1}$, E_8 is the E_8 lattice, and $II_{1,1}$ is the 2-dimensional even indefinite unimodular lattice. If L is any lattice and n is an integer then $L(n)$ means L with the bilinear form multiplied by n. We write vectors of $M = L \oplus II_{1,1}(2)$ as (v, m, n) with $v \in L$, $m, n \in \mathbf{Z}$, so that this vector has norm $v^2 + 4mn$. Similarly we write vectors of $L = E_8(-2) \oplus II_{1,1}$ as (v, m, n) with $v \in E_8(-2)$, $m, n \in \mathbf{Z}$, so that this vector has norm $v^2 + 2mn$. The norm 0 vector u of M is defined to be the vector $(0,0,1) \in M$, and similarly we define $\rho = (0,0,1) \in L$ and $\rho' = (0,1,0) \in L$. We write $O_M(\mathbf{Z})$ for the automorphism group of the lattice M. We will say that a vector of a lattice has even type if it has even inner product with all vectors, and we will say it has odd type otherwise. There are two orbits of primitive norm 0 vectors of M under $O_M(\mathbf{Z})$, which can be distinguished by whether they have even or odd type. There are also two orbits of primitive norm 0 vectors in L which can be distinguished in the same way.

We define the hermitian symmetric space Ω of M to be the set of vectors $\omega \in P(M \otimes \mathbf{C})$ such that $(\omega, \omega) = 0$ and $(\omega, \bar{\omega}) > 0$ (where P means the projective space of a vector space). The space Ω has two components, and we write $O_M(\mathbf{Z})^+$ for the subgroup of index 2 of $O_M(\mathbf{Z})$ of elements that do not exchange these two components. There is a second model for Ω which we will use in Section 3. The positive norm vectors of $L \otimes \mathbf{R}$ form two open cones, and we choose one of them and call it C. Then one of the two components of Ω can be identified with $L \otimes \mathbf{R} + iC$ by identifying the point $v \in L \otimes \mathbf{R} + iC$ with the point of Ω represented by $(v, 1/2, -v^2/2) \in M \otimes \mathbf{C}$.

Any automorphism of $M \otimes \mathbf{R}$ induces an automorphism of Ω, and if it does not exchange the two components of Ω this induces an automorphism of $L \otimes \mathbf{R} + iC$. We describe these automorphisms in a few cases which we will need later. If σ is an automorphism of $M = L \oplus II_{1,1}(2)$ fixing all vectors of $II_{1,1}$ then it induces an automorphism of L and hence of $L \otimes \mathbf{R} + iC$ in the obvious way. If $\lambda \in L'$ then there is an automorphism of M taking (v, m, n) to $(v + 2m\lambda, m, n - (v, \lambda) - m\lambda^2)$, and the induced automorphism of $L \otimes \mathbf{R} + iC$ takes y to $y + \lambda$. Finally, reflection in the hyperplane of the vector $(0, 1, -1/2)$ of $M \otimes \mathbf{Q}$ takes $(v, 1/2, -v^2/2)$ to $(v, -v^2, 1/4)$ and therefore induces the automorphism of $L \otimes \mathbf{R} + iC$ taking y to $-y/2(y, y)$.

Remarks. The group $O_M(\mathbf{Z})$ seems to be defined slightly differently from the group Γ in [1] but it follows from [6, Remark 1.15] that these two groups are the same. Similarly Proposition VIII.20.6 of [1] states that there are finitely many equivalence classes of norm -2 vectors of M under $O_M(\mathbf{Z})$, but it follows from [6] (or from Lemma 2.3) that there is in fact only one orbit of norm -2 vectors. In particular, the divisor $\bigcup_d H_d/O_M(\mathbf{Z})$ on $\Omega/O_M(\mathbf{Z})$ is irreducible, and not just a finite union of irreducible divisors.

If d is a norm -2 vector of M we write H_d for the divisor of points of Ω represented by points orthogonal to d. Then it follows from 20.5, 21.2, 21.4, of [1, VIII], or from [6, 1.14] that the moduli space of Enriques surfaces is

$$D^0 = \left(\Omega \Big\backslash \left(\bigcup_d H_d\right)\right) \Big/ O_M(\mathbf{Z}).$$

In the rest of this section we prove some auxiliary results about the subgroups of $O_M(\mathbf{Z})$ generated by various subsets.

LEMMA 2.1. *Suppose that v is a vector in $L \otimes \mathbf{Q}$ but not in the dual L' of L, and suppose that x is any real number. Then we can find a vector $\mu \in L$ with $\mu^2 \equiv 2 \bmod 4$ such that $|(\mu - v)^2 - x| < 2$.*

Proof. As v is not in L' we can find a primitive norm 0 vector ρ of L of odd type such that (ρ, v) is not an integer. This is because the primitive norm 0 vectors of odd-type span L. As $O_L(\mathbf{Z})$ acts transitively on such norm 0 vectors we can assume that $\rho = (0,0,1) \in E_8(-2) \oplus II_{1,1}$. Then $v = (\lambda, a, b)$ with a not an integer. We will find some μ of the form $\mu = (0, m, n) \in II_{1,1}$, with m and n both odd so that $\mu^2 = 2 \bmod 4$. So we have to find odd integers m and n satisfying $|(\mu - v)^2 - x| = |\lambda^2 + 2(a - m)(b - n) - x| < 2$. As a is not an integer we can find some odd m with $0 < |a - m| < 1$. Then whenever we add 2 to n we change $2(a - m)(b - n)$ by a non-zero number less than 4, so we can choose some odd integer n so that $2(a - m)(b - n)$ is at a distance of less than 2 from any given real number $x - \lambda^2$. This proves Lemma 2.1. □

LEMMA 2.2. *Suppose that R_2 is the set of norm -2 vectors of M having inner product 2 with $u = (0,0,1)$, and Γ_1 is the subgroup of $O_M(\mathbf{Z})$ generated by the reflections of vectors of R_2 and the automorphism -1. Then any vector $r \in M$ is conjugate under Γ_1 to a vector of the form $(v, m, n) \in M$ with either $m = 0$ or $v/m \in L'$ and $m > 0$.*

Proof. We can assume that $r = (v, m, n)$ has the property that $|(r, u)| = |2m|$ is minimal among all conjugates of r under Γ_1. If $m = 0$ we are done, so we can assume that $m \neq 0$, and we wish to show that $v/m \in L'$.

Suppose that $v/m \notin L'$. By Lemma 2.1 we can find a vector $\mu \in L$ with $|(\mu - v/m)^2 + (-4n/m - v^2/m^2)| < 2$ and $\mu^2 \equiv 2 \bmod 4$. But if we calculate the inner product (r', u), where r' is the reflection or r in the hyperplane of $(\mu, 1, (-\mu^2 - 2)/4) \in R_2$, we find that (r', u) has absolute value $|(r', u)| = |(r, u + 2(\mu, 1, (-\mu^2 - 2)/4))| = |m((\mu - v/m)^2 + (-4n/m - v^2/m^2))| < |2m| = |(r, u)|$, which is not possible because we assumed that $|(r, u)| = |2m|$ was minimal. Hence $v/m \in L'$. We can obviously then assume that $m > 0$ by using the automorphism -1. This proves Lemma 2.2. □

We write R_0 or R_2 for the sets of norm -2 vectors of M which have inner products 0 or 2 with u. We let Γ_1 be the group generated by -1 and by the reflections of elements of R_2, and we let Γ_2 be the group generated by -1 and the reflections of elements of $R_0 \cup R_2$.

LEMMA 2.3. *Any norm -2 vector of M is conjugate to an element of $R_0 \cup R_2$ under the group Γ_1. In particular, the group Γ_2 is the group generated by -1 and the reflections of all norm -2 vectors of M.*

Proof. Put $r = (v, m, n)$, so that by Lemma 2.2 we can assume that either $m = 0$ or $v/m \in L'$ and $m > 0$. If $m = 0$ then r is orthogonal to u so this case is trivial.

Now suppose that $v/m \in L'$ (so that $(v/m, v/m) \in \mathbf{Z}$) and $m > 0$. Then $-2 = (r, r) = m^2(v/m, v/m) + 4mn$ is divisible by $(m^2, 4m)$, so $m = 1$. Hence $r = (v, 1, n)$ is an element of R_2. This proves Lemma 2.3. □

Remark. Lemma 2.3 can be used to give an elementary proof of Namikawa's result [6, 2.13] that $O_M(\mathbf{Z})$ acts transitively on the norm -2 vectors of M, which implies that the complement of D^0 in D is an irreducible divisor. This can be done by noting that R_0 and R_2 are acted on transitively by the groups in (1) and (2) of Lemma 2.4, and then showing that there is some automorphism of M (e.g., reflection in a norm -4 vector) mapping some vector of R_0 into R_2.

LEMMA 2.4. *Two primitive norm* 0 *vectors* z *and* z' *of* M *of even type are in the same orbit of* Γ_2 *if and only if they are congruent* mod $2M$. *In particular, there are only a finite number of orbits of such norm* 0 *vectors under* Γ_2.

Proof. By Lemma 2.3 the group Γ_2 is the group generated by -1 and all reflections of norm -2 vectors and hence is normal in $O_M(\mathbf{Z})$. We can find a norm 0 vector $u \in 2M'$ such that $(u, z) \not\equiv 0 \bmod 4$ because z is primitive, and as Γ_2 is normal in $O_M(\mathbf{Z})$ we may assume that $u = (0, 0, 1)$. Any conjugate of z under Γ_2 is congruent to $z \bmod 2M$ and therefore has inner product with u not divisible by 4, and in particular is not orthogonal to u. By Lemma 2.2 this implies that we may assume that $z = (mv, m, n)$ for some $v \in L'$, $m > 0$. As z has norm 0, we see that $m^2v^2 = -4mn$, so $mv^2 = -4n$ as $m \neq 0$. The vector $2v$ is in L, so if m is divisible by some odd number p then $z/p \in M$. As $m = (z, u)/2$ is odd and z is primitive this shows that $m = 1$.

Hence we can assume that $z = (v, 1, n)$ and $z' = (v', 1, n')$ with $v \equiv v' \bmod 2L$. If r is a norm -2 vector of L then the products of the reflections of $(r, 0, 0)$ and $(r, 0, 1)$ is the automorphism taking $(v, 1, n)$ to $(v + 2r, 1, n - (v, r) + 2)$. The lattice L is generated by its norm -2 vectors r, so these automorphisms can be used to map z to z'. This proves Lemma 2.4. □

LEMMA 2.5. *The group* Γ_3 *generated by the following sets of automorphisms has finite index in* $O_M(\mathbf{Z})^+$.

(1) *The automorphisms in* $O_L(\mathbf{Z})^+$ *(extended to automorphisms of* M *by letting them act trivially on* $II_{1,1}(2)$*).*
(2) *The group of automorphisms taking* (v, m, n) *to* $(v + 2m\lambda, m, n - (v, \lambda) - m\lambda^2)$ *for* $\lambda \in L'$. *This is the group of all automorphisms of* M *fixing* u *and all vectors of* $M/\langle u\rangle$.
(3) *An automorphism given by reflection of norm* -2 *vector* r *of* M *which has inner product* 2 *with* u. *(The group in* (2) *above acts transitively on the set of such norm* -2 *vectors* r*, so it does not matter which we choose.)*
(4) *The automorphism* -1.

Proof. The group generated by the automorphisms in (1) and (2) above is the group of all automorphisms in $O_M(\mathbf{Z})^+$ that fix the primitive norm 0 vector u of even type, so to prove that Γ_3 has finite index in $O_M(\mathbf{Z})^+$ is is sufficient to show that there are only a finite number of orbits under Γ_3 of primitive norm 0 vectors of even type. But Γ_3 contains Γ_2 because the group of automorphisms in (2) acts transitively on the set of norm -2 vectors having inner product 2 with u and the group of automorphisms in (1) contains all reflections of vectors of R_0, and by Lemma 2.4 Γ_2 has only a finite number of orbits on the set of primitive norm 0 vectors of even type. This proves Lemma 2.5. □

Unfortunately, the group Γ_3 generated by the transformations above is not the whole group $O_M(\mathbf{Z})^+$, and this means that the proof that Φ is an automorphic form for $O_M(\mathbf{Z})^+$ in Section 3 has to be indirect. For example, the transformations above all preserve the set of vectors in M of even type whose inner product with u is not divisible by 4, and it is easy to see that u is not in this set but is conjugate to a vector in this set under reflection in a norm -4 vector having inner product -2 with u.

3. CONSTRUCTION OF THE AUTOMORPHIC FORM Φ

In this section we construct the automorphic form Φ of weight 4 for $O_M(\mathbf{Z})^+$ on one of the two components of Ω whose zeros are exactly the divisors orthogonal to norm -2 vectors of M. We will construct Φ as a function on $L \otimes \mathbf{R} + iC$.

We start by recalling from [2, Section 14, Example 3] the twisted denominator formula for an automorphism of the monster Lie algebra coming from an involution of the Leech lattice with an 8-dimensional fixed subspace and using it to define Φ. Unfortunately, there are 2 misprints the formulas given there: the final term in the first formula on page 442 should be $(-1)^{m+n}|p_g((1-r^2)/2)|$, and the factor $q^{1/2}$ in the next line should not be there. It should also be noted that the sign conventions for Lorentzian lattices in [2] are the opposite to those used here. With these changes the twisted denominator formula is

$$\Phi(y) = \sum_{w \in W} \det(w) e^{2\pi i(\rho, w(y))} \prod_{n>0} (1 - e^{2\pi i n(\rho, w(y))})^{(-1)^n 8}$$

$$= e^{2\pi i(\rho, y)} \prod_{r \in \Pi^+} (1 - e^{2\pi i(r, y)})^{(-1)^{(r, \rho - \rho')} c((r,r)/2)}$$

where the first equality is the definition of Φ and the second equality only holds in the region of convergence of the infinite product (see the remark after Lemma 3.1). The vector y is an element of $L \otimes \mathbf{C}$ with $\mathcal{I}(y) \in C$, where C is the positive open cone in $L \otimes \mathbf{R}$. The group W is the subgroup of $O_L(\mathbf{Z})$ generated by the reflections of the norm -2 vectors of L (and has infinite index in the full reflection group of L). It is also the Weyl group of the fake monster Lie superalgebra. The vectors ρ and ρ' are the norm zero vectors $(0,0,1)$ and $(0,1,0)$ of $L = E_8(-2) \oplus II_{1,1}$, and ρ is also the Weyl vector of the fake monster Lie superalgebra. The set Π^+ is the set of positive roots of the fake monster Lie superalgebra, which consists of all nonzero vectors (v, m, n) of norm at least -2 such that $m > 0$ or $m = 0$ and $n > 0$. The numbers $c(n)$ are the coefficients of

$$\begin{aligned} f(\tau) &= \sum_n c(n) q^n \\ &= \eta(\tau)^{-8} \eta(2\tau)^8 \eta(4\tau)^{-8} \\ &= q^{-1}(1+q)^8(1-q^2)^{-8}(1+q^3)^8 \cdots \\ &= q^{-1} + 8 + 36q + 128q^2 + 402q^3 + 1152q^4 + 3064q^5 + O(q^6). \end{aligned}$$

LEMMA 3.1. *The sequence* $\log(c(n))$ *is asymptotic to* $2\pi\sqrt{n}$.

Proof. The circle method (see [7]) gives an asymptotic expansion for the coefficients $c(n)$ of any meromorphic modular form of negative weight with no poles in the upper half plane in terms of the poles at cusps. The dominant term in this asymptotic expansion for $f(\tau)$ comes from the pole of order 1/4 of $f(\tau)$ at the cusp 0, given by $\tau^4 f(-1/\tau) = 16\eta(\tau)^{-8}\eta(\tau/2)^8\eta(\tau/4)^{-8} = 16q^{-1/4} + \cdots$. This shows that $c(n)$ is asymptotic to $\pi n^{-5/2} I_5(2\pi\sqrt{n})$, where I_5 is a modified Bessel function (see [3, Lemma 5.3]). As $I_5(x)$ is asymptotic to $e^x/\sqrt{2\pi x}$ we get the result stated in the lemma by taking logs. This proves Lemma 3.1. □

In particular, this implies that the infinite product for Φ converges whenever y has an imaginary part in the open region bounded by the hypersurface S of points $y \in iC$ with $(y, y) = -1/2$ (see the proof of Lemma 3.3).

Remark. It is easy to check Lemma 3.1 by computing a few cases numerically; for example, the values of $\pi n^{-5/2} I_5(2\pi\sqrt{n})$ for $n = -1, 0, 1, 2, 3, 4, 5$ are 1.17, 8.01, 35.59, 128.02, 402.80, 1151.95, and 3062.48, which can be compared with the values of the $c(n)$'s given above.

THEOREM 3.2. *The function $\Phi(y)$ is an automorphic form on $L \otimes \mathbf{R} + \mathrm{i}C$ with respect to the discrete subgroup $O_M(\mathbf{Z})^+$ and the character χ of $O_M(\mathbf{Z})^+$ (defined below).*

The proof of this theorem will take most of the rest of this section. We first note the following two obvious transformation laws for Φ, which follow immediately from the definition of Φ. If $\sigma \in O_L(\mathbf{Z})^+$ then

$$\Phi(\sigma(y)) = \chi(\sigma)\Phi(y)$$

where χ is a character of $O_L(\mathbf{Z})^+$ taking reflections of norm -2 vectors to -1 and taking reflections of norm -4 vectors to 1. If $\lambda \in L'$ then

$$\Phi(y + \lambda) = \Phi(y).$$

The next lemma is essentially a special case of Theorem 5.1 of [3].

LEMMA 3.3. *If we define $\Psi(y) = \Phi(y + (\rho - \rho')/2)$ then $\Psi(y)$ vanishes whenever y lies on the surface $S \subset \mathrm{i}C$ of points $y_0 \in \mathrm{i}C$ with $(y_0, y_0) = -1/2$.*

Proof. We can assume that $y_0 \in \mathrm{i}(L \otimes \mathbf{Q})$ because rational points are dense in S. If y_0 is a point in $S \cap \mathrm{i}(L \otimes \mathbf{Q})$ then we look at the function $g(\tau) = -\log(\Psi(\tau y_0/\mathrm{i})\exp(-2\pi\tau(\rho, y_0))$, defined for $\mathscr{I}(\tau)$ large. This can be expanded as a power series $g(\tau) = \sum_{n \in \mathbf{Q}} a(n)q^n$ in some rational power of $q = e^{2\pi i\tau}$.

First we show that the coefficients of g are non-negative. If we look at the infinite product expansion

$$\Psi(y) = -e^{2\pi i(\rho, y)} \prod_{r \in \Pi^+} (1 - (-1)^{(r, \rho - \rho')} e^{2\pi i(r, y)})^{(-1)^{(r, \rho - \rho')} c((r, r)/2)}$$

we can see that all the Fourier coefficients $a(r)$ of $-\log(-\Psi(y)/e^{2\pi i(\rho, y)}) = \sum_r a(r) e^{2\pi i(r, y)}$ are non-negative. This is because if r is a primitive vector then the Fourier coefficients $a(nr)$ $(n > 0)$ are given by the Fourier coefficients of

$$-\log \prod_{m > 0} (1 - e^{2\pi i m(r, y)})^{c((mr, mr)/2)}$$

if $(r, \rho - \rho')$ is even, and by

$$-\log \prod_{m > 0} (1 - (-1)^m e^{2\pi i m(r, y)})^{(-1)^m c((mr, mr)/2)} = -\log \prod_{m > 0, m \text{ odd}} (1 - e^{2\pi i m(r, y)})^{c((mr, mr)/2)}$$

$$-\log \prod_{m > 0, m \text{ even}} (1 - e^{2\pi i m(r, y)})^{c((mr, mr)/2) - c((mr/2, mr/2)/2)}$$

if $(r, \rho - \rho')$ is odd, and in both cases we see that all the Fourier coefficients are non-negative because $0 \leqslant c(n) \leqslant c(4n)$ for any n. This implies that the coefficients in the series for g are non-negative because g is the restriction of $-\log(-\Psi(y)/e^{2\pi i(\rho, y)})$ to a line so that its coefficient $a(n)$ are given by $a(n) = \sum_{(r, y_0) = n} a(r)$.

By using Lemma 3.1 we can check that $\limsup_{n\to+\infty}\log(a(n))/n = 2\pi$. We get 2π as an upper bound for the lim sup because in the sum $a(n) = \sum_{(r,y_0)=n} a(r)$ the number of terms is bounded by a polynomial (of degree 9) in n, and in each term (r, r) is at most $n^2/|y_0^2| = 2n^2$, and $a(r)$ is not much bigger than $c((r, r)/2)$, whose log is about $2\pi\sqrt{(r, r)/2} \leqslant 2\pi n$ by Lemma 3.1. We can prove that 2π is a lower bound for the lim sup in a similar way, by observing that all the coefficients $a(r)$ are positive so that $a(n)$ is bounded below by the largest of them, and that there are infinitely many n such that the largest $a(r)$ in the sum is very roughly $e^{2\pi\sqrt{(n^2/|y_0^2|)/2}} = e^{2\pi n}$. Therefore the series for g has radius of convergence $|q| = \lim\sup|a(n)|^{-1/n} = e^{-2\pi}$.

Hence g has a singularity at $e^{2\pi i\tau} = q = e^{-2\pi}$ because a power series with non negative coefficients with radius of convergence $e^{-2\pi}$ has a singularity at $e^{-2\pi}$. (This is why we have to replace Φ by Ψ: the coefficients of $-\log(\Phi(y)/e^{2\pi i(\rho, y)})$ do not all have the same sign.) This means that $g(\tau) = -\log(\Psi(\tau y_0/i)\exp(-2\pi\tau(\rho, y_0)))$ has a singularity at $\tau = i$, so that $\log(\Psi(y))$ has a singularity at $y = y_0$. However $\Psi(y)$ is holomorphic at $y = y_0$, so the only way that $\log(\Psi(y))$ can have a singularity at $y = y_0$ is if Ψ vanishes at y_0. This shows that Ψ vanishes on the surface S and proves Lemma 3.3.

The main step in the proof of Theorem 3.2 is the proof of the following extra transformation law for Φ (or rather for Ψ).

LEMMA 3.4.

$$\Psi(-y/2(y, y)) = -16(y, y)^4\Psi(y).$$

Proof. It is sufficient to prove this for purely imaginary values of y because then the result is true for all y by analytic continuation. The cone iC has a pseudo-Riemannian metric induced by the bilinear form on $L\otimes\mathbf{R}$ and has an associated wave operator given by the Laplacian of its pseudo-Riemannian metric. On the space iC, Ψ is a solution of the wave equation because each of the terms $\exp((w(n\rho), y)$ in the sum defining Φ is a solution of the wave equation (as each of the vectors $w(n\rho)$ has norm 0). This implies that $(y, y)^{10/2-1}\Psi(-y/16(y, y))$ is also a solution of the wave equation by the transformation of the wave operator under the conformal transformation $y \to -y/2(y, y)$ of iC. (For this special conformal transformation this is easy to check directly as it is just the fact that if $\Psi(y)$ is any solution to the wave equation in n dimensions then so is $(y, y)^{n/2-1}\Psi(-y/c(y, y))$ for any positive constant c. The quickest way to prove this is to choose orthogonal coordinates so that $y = (x_1, \ldots, x_n)$ and calculate $(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2} - \frac{\partial^2}{\partial x_n^2})(y, y)^{n/2-1}\Psi(-y/c(y, y))$ explicitly to show that it vanishes at y. Because of the invariance of everything under Lorentzian transformations, it is only necessary to check vanishing at points y of the form $(0, \ldots, 0, r)$.)

Now we check that that $-16(y, y)^4\Psi(-y/2(y, y))$ and $\Psi(y)$ both have the same partial derivatives of order at most 1 on the surface S. They both vanish on S by Lemma 3.3 and therefore have the same constant term, and for the same reason their first partial derivatives in any direction tangent to S both vanish, so it is only necessary to check that they have the same first partial derivatives in the direction normal to S. But if Ψ is any smooth function on iC whose partial derivative normal to S at a point $s \in S$ is x, then $\Psi(-y/2(y, y))$ has a partial derivative normal to S at s of $-x$. (This follows by restricting Ψ to the line through 0 and s and using the elementary fact that if a differentiable function h is defined for positive reals y then the derivatives of $h(y)$ and $-h(y/2y^2) = -h(1/2y)$ are equal at $y = 1/\sqrt{2}$.) The function $-2(y, y)$ is 1 on S, so the partial derivative normal to S of $(-2(y, y))^n\Psi(-y/2(y, y)$ is $-x$ for any integer n because $\Psi(-y/2(y, y))$ vanishes for

$y \in S$. Hence $-16(y,y)^4\Psi(-y/2(y,y))$ and $\Psi(y)$ both have the same partial derivatives of order at most 1 on the surface S. These two functions both satisfy the wave equation and have the same partial derivatives of order at most 1 on the non characteristic surface S, so by the uniqueness part of the Cauchy–Kovalevsky theorem they must be equal on iC. This proves Lemma 3.4. □

LEMMA 3.5. *Φ is an automorphic form of weight 4 for the finite index subgroup Γ_3 of $O_M(\mathbf{Z})$ (defined in Lemma 2.5).*

Proof. Lemma 3.4 shows that Ψ transforms like an automorphic form under reflection in the norm -2 vector $(0,1,-1/2)$ of $M \otimes \mathbf{Q}$ (which is not in M). We obtain Ψ from Φ by applying the automorphism of $M \otimes \mathbf{Q}$ taking (v,m,n) to $(v+2\lambda m, m, n-(v,\lambda)-\lambda^2 m)$, where $\lambda = (\rho-\rho')/2 \in L \otimes \mathbf{Q}$. This automorphism takes $(0,1,-1/2)$ to the norm -2 vector $(\rho-\rho',1,0) \in L$. Hence Φ transforms like an automorphic form under reflection in the norm -2 vector $(\rho-\rho',1,0)$ having inner product 2 with u. We have therefore verified that Φ transforms an automorphic form under all the transformations of Lemma 2.5, which proves Lemma 3.5. □

LEMMA 3.6. *The form Φ vanishes (to order 1) along all the divisors of norm -2 vectors of M.*

Proof. We know by Lemma 2.3 that any norm 2 vector is conjugate to a vector in either R_0 or R_2, and the groups in (1) and (2) of Lemma 2.5 act transitively on these two sets, so it is sufficient to prove that Φ vanishes along the divisors of one vector in R_0 and one vector in R_2. But Φ vanishes along the divisor of the vector $(\rho-\rho',1,0) \in R_2$ by Lemma 3.3 (see the proof of Lemma 3.5), and Φ vanishes along the hyperplane orthogonal to any vector in R_0 because the functional equation $\Phi(\sigma(y)) = \chi(\sigma)\Phi(y)$ implies that Φ changes sign under reflection in this hyperplane. This proves Lemma 3.6. □

LEMMA 3.7. *There exists an automorphic form of weight $16632/2$ for the lattice $II_{2,10} = E_8(-1) \oplus II_{1,1} \oplus II_{1,1}$ whose zeros are exactly the hyperplanes of norm -4 vectors of $II_{2,10}$.*

Proof. The meromorphic modular form

$$E_4(\tau)^5/\Delta(\tau)^2 - 1248E_4(\tau)^2/\Delta(\tau) = q^{-2} + 16632 + O(q)$$

has weight -4 and level 1 and no poles on the upper half plane, so applying Theorem 10.1 of [3] shows the existence of an automorphic form with the required properties. This proves Lemma 3.7.

LEMMA 3.8. *The only zeros of Φ lie on the hyperplanes of norm -2 vectors of M.*

Proof. The group $II_{2,10}/2II_{2,10}$ has order 2^{12} and its elements have a well defined norm mod 4. Under the group $O_{II_{2,10}}(\mathbf{Z})$ its elements split into 3 orbits: the zero element, an orbit of size $2^{11}-2^5 = 2016$ of elements of norm congruent to 2 mod 4, and an orbit of size $2^{11}+2^5-1 = 2079$ of non-zero elements whose norm is congruent to 0 mod 4. We note that every norm -4 vector v of $II_{2,10}$ gives a unique norm 0 mod 4 non-zero element v of $II_{2,10}/2II_{2,10}$, and this partitions the norm -4 vectors of $II_{2,10}$ into 2079 disjoint classes.

For each of the 2079 non-zero vectors of norm $0 \bmod 4$ in $II_{2,10}/2II_{2,10}$, an inverse image of this vector in $II_{2,10}$ together with $2II_{2,10}$ generates a copy of $M(2)$. For each of these 2079 copies of $M(2)$ we take a copy of the form Φ corresponding to it (with its argument rescaled by a factor of $\sqrt{2}$) and we multiply these 2079 automorphic forms together to get a function Θ. (It is not yet clear that Θ is uniquely defined by this, because we have not yet proved that Φ is an automorphic form for the whole of $O_M(\mathbf{Z})^+$, but this does not matter.) By Lemma 3.5 Θ is an automorphic form for some finite index subgroup of $O_{II_{2,10}}(\mathbf{Z})$ of weight 4×2079. The hyperplane of any norm -4 vector v of $II_{2,10}$ is a zero of the factor of Φ corresponding to the vector $v \in II_{2,10}/2II_{2,10}$ (which corresponds to a norm -2 vector in the copy of M), so Θ vanishes on all the hyperplanes of all norm -4 vectors of $II_{2,10}$. Therefore, we can divide Θ by the automorphic form of Lemma 3.7 to obtain an automorphic form of weight $4 \times 2079 - 16632/2 = 0$ which is holomorphic at cusps by the Koecher boundedness principle. This quotient must therefore be a constant, so it has no zeros, and therefore the form Θ has no zeros other than those corresponding to norm -4 vectors of $II_{2,10}$. But this implies that Φ has no zeros other than those corresponding to norm -2 vectors of M, otherwise these would give rise to other zeros of Θ. This proves Lemma 3.8. □

We can now complete the proof of Theorem 3.2. By the Koecher boundedness principle an automorphic form on Ω is determined up to multiplication by a constant by its zeros on Ω, because if f and g are two forms with the same zeros then f/g and its inverse g/f are both automorphic forms so they must both be constant. The transform of Φ under any element of $O_M(\mathbf{Z})^+$ has the same zeros as Φ because the zeros of Φ just correspond to the norm -2 vectors of M. Hence the transform of Φ under any element of $O_M(\mathbf{Z})$ is equal to Φ multiplied by some nonzero constant. This proves Theorem 3.2. □

Remark. The proof that Φ has no extra zeros relies on a strange numerical coincidence. If we assume that Φ is an automorphic form for the group $O_M(\mathbf{Z})^+$ then we can give a more conceptual proof of this as follows. (Unfortunately, the proof that Φ is an automorphic form for $O_M(\mathbf{Z})^+$ uses the fact that Φ has no extra zeros, so this argument is of no use unless someone finds a different proof that Φ is automorphic under $O_M(\mathbf{Z})^+$!) By Theorem 5.1 of [3] any zero of Φ must be the hyperplane of some primitive vector v of M (which is a subset of the hermitian symmetric space called a rational quadratic divisor in [3]). We have to prove that v has norm -2. By Lemma 3.9 below there is some primitive norm 0 vector orthogonal to v. As Φ transforms like an automorphic form under $O_M(\mathbf{Z})^+$, we can assume that this is the norm zero vector u. But then the divisor of v intersects the region of convergence of the infinite product defining Φ, which is only possible if it is a zero of one of the factors in the infinite product. But the only factors in the infinite product with zeros are those of the form $1 - \exp(2\pi i(x, y))$ with x a vector of norm -2. This shows that v is a vector of norm -2 and hence shows that the zeros of Φ are exactly the hyperplanes of norm -2 vectors of M.

LEMMA 3.9. *Any vector of M is orthogonal to a conjugate of u under $O_M(\mathbf{Z})$.*

Proof. The lattice M contains a 2-dimensional primitive isotropic sublattice U such that every vector in U has even type, so that every primitive vector in U is conjugate to u under $O_M(\mathbf{Z})^+$. As U has dimension greater than 1, there is some primitive vector in U orthogonal to v, which has the required properties. This proves Lemma 3.9. □

Remark. It is not true that any vector is conjugate under the group Γ_3 to a vector orthogonal to u; for example, this is not true for a vector of even-type having inner product with u not divisible by 4. It is also not true that any vector v of M is orthogonal to a primitive isotropic vector u of odd type. In fact, it is not hard to check that if v has this property then v has norm (v, v) divisible by 4. The proof of Lemma 3.9 breaks down for this case because lattices U in the other orbit of 2-dimensional primitive isotropic sublattices still have some primitive vectors of even type.

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